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Product Innovation of an Incumbent Firm: A Dynamic Analysis*

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Abstract

In case of a product innovation firms start producing a new product. While doing so, such a firm should decide what to do with its existing product after the firm has innovated. Essentially it can choose

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between replacing the established product by the new one, or keep on producing the established product so that it produces two products at the same time.

The aim of this paper is to design a theoretical framework to analyze this problem. Due to technological progress the quality of the newest available technology, and thus the quality of the innovative product that can be produced by this technology, increases over time. The implication is that a later innovation enables the firm to produce a better innovative product. So, typically the firm faces the tradeoff between innovating fast, which boosts its profits soon but only by a small amount, or innovating later, which leads to a larger payoff increase. The drawback here is that the firm is stuck with producing the established product for a longer time.

We find that a highly uncertain economic environment makes the firm delay abolishing the old product market. But if the innovative market is more volatile, the firm enters the market sooner, provided it will be active on the old market, at least for some time. Moreover, the smaller the initial demand for the innovative product market, the better the quality of the innovative product needs to be for the product innovation to be optimal.

1 Introduction

Technological progress implies that, as time passes, better technologies and therewith, also better products, appear on the market. One of the questions is then: when should a firm invest in such technologies and products, and what should the firm do with the existing ones? Should the firm keep producing old products, or replace the old product when introducing a more innovative one? Essentially it can choose between replacing the established production process by the new one (single rollover), or keep on producing the established product so that it produces two products at the same time (dual rollover). The advantage of the latter is that the firm earns revenue from both markets, but, if the innovative product is a strategic substitute to the established product, the firm is competing with itself in the sense that growth of the innovative product market will attract consumers that at the same time leave the established product market, or the other way around. Also after initially choosing to simultaneously produce the established and the innovative product, after some time it can be optimal to stop taking the established product into production due to the just described cannibalization effects.

The aim of this paper is to design a theoretical framework to analyze this problem. We start out with a firm producing an established product. The firm has an option to carry out a product innovation. To do so it has to adopt a new technology by which it can produce the innovative product. New technologies become available as time passes. Due to technological progress the quality of the newest available technology, and thus the quality of the innovative product that can be produced by this technology, increases over time, albeit in a stochastic way, because the firm does not know beforehand how fast technologies will develop. To capture this we impose that the technological progress is modelled by a Poisson process, where at discrete moments in time the quality jumps upward. The implication is that a later innovation enables the firm to produce a better innovative product, which will stimulate the innovative product demand and thus the innovative product revenue. So, typically the firm faces the tradeoff between innovating fast that enlarges its payoff soon but only by a small amount, or innovating later that leads to a larger payoff increase, the drawback being that the firm is stuck with producing the established product for a longer time.

While perfectly being aware of the size of the demand of the established product, the firm does not know beforehand how consumers will appreciate the innovative product and thus how demand of this product will develop over time. Therefore, we assume that demand of the new product is also stochastic, such that the output price satisfies a geometric Brownian motion (GBM) process. A change in demand on the new market directly influences the size of the cannibalization effect on the established market, so therefore we impose that this cannibalization effect is also subject to the same GBM process.

Except from determining the optimal time to innovate, we also analyze the choice between the "add" and the "replace" option, where the replace option reflects the possibility that at the innovation time the firm stops producing the established product and begins producing the innovative product. The add option means that after innovating the firm produces both the established and the innovative product. After deciding to produce both products, the firm still has the option to stop producing the established product, which will boost demand of the innovative product.

We now explain in what way we extend the existing literature. Farzin et al. [1998], Doraszelski [2001] and Doraszelski [2004] focus on the time to innovate where technological progress develops stochastically over time. The expected rate of new technologies arriving over time is constant, an assumption which is relaxed in Hagspiel et al. [2015]. Cho and McCardle [2009] consider a firm simultaneously using two types of technologies and analyze the effect of their interdepencies on the timing of adopting upgrades. Smith and Ulu [2012] allow for uncertainty in future costs of adoption. Murto [2007] considers the effect of revenue uncertainty. Our paper also takes revenue uncertainty into account, but in addition to the just mentioned papers we explicitly analyze how to go further with the established product market after innovating.

In Grenadier and Weiss [1997] different innovation strategies are outlined that take into account the established technology, but to innovate or not is just a yes-or-no decision. The new technology has given characteristics, so it is not taken into account that the newest technologies improve as time passes, as we do.

Reinganum [1981], Fudenberg and Tirole [1985], and Milliou and Petrakis [2011] determine the optimal time to innovate in a framework where two firms have this innovation option, but their models are deterministic. Another difference with our work is that a process innovation that reduces costs is considered instead of a product innovation. Stenbacka and Tombak [1994] add a new element to the literature on the timing of adoption by explicitly taking the uncertainty in the length of time required for successful implementation into consideration.

Huisman and Kort [2004] present a model with two firms that both can choose between adopting an existing technology immediately or wait for a given new technology that is better than the old one, which becomes available at some future unknown point in time.

Our work is closely related with Kwon [2010] and Hagspiel et al. [2016a], in the sense that we also study the option to invest in a new product to boost the firm's profit. But here we consider some remarkable extensions. First, we assume that innovations occur according to a jump process, and therefore we have, besides the (stochastic) price, the state of technology. The former authors also address the option to invest in a new product, but there are no innovation events, being the price as the only stochastic variable. Moreover, we include the option to produce both kinds of products until it is no longer optimal, and therefore the established product stops being produced. We show that the firm has more incentive to first add the innovative product to the product portfolio if demand volatility for the new product is high. In fact, the option to replace increases with demand volatility. Therefore, the firm keeps this option alive for a longer time if demand for the new product is more uncertain. If the interest rate is large, however, the firm is more inclined to replace the old product immediately.

Accounting for the option to eventually replace the old product, we find that the decision whether to add the new product to the product portfolio or replace the old product immediately, has a crucial effect on the innovation decision. We show that if the innovative market is more volatile, the firm enters the market sooner, provided it will be active on the old market, at least for some time. This is due to the replace option the firm gains after deciding to innovate, which increases in uncertainty, and therefore leads to a higher value of investing. Furthermore, the smaller the initial demand for the innovative product, the better the quality of the new product needs to be before it is optimal to innovate.

The remaining paper is organized as follows. Section 2 introduces the model. In Section 3 we derive the optimal innovation policy and show the optimal time to replace the old product by new one. In this section we also introduce a benchmark case, where the demand of the innovative product is assumed to be deterministic, which serves us as reference point in the comparative statics presented in Section 4. Section 5 concludes.

2 Model

Our model will consider a market for established and innovated products. The research questions we want to deal with are, first, to establish the optimal time to innovate of the firm. Second, after the firm has innovated, how long, if at all, the firm should be active on the established product market. In order to do so, we keep our framework as simple as possible. Among others this requires that capacity sizes for the old and the new product are treated as parameters. Later on, on top of our analytical comparative statics results, we provide a numerical illustration showing how results determine on different capacity sizes of both products.

We consider an incumbent firm that is currently producing an established product with capacity K_0. The firm produces up to capacity[1]. The price for the established product satisfies

$$p_0 = \xi_0 - \alpha K_0,$$

assuming that $\xi_0 > \alpha K_0$, where ξ_0 is the maximum willingness to pay for the established product and $\alpha > 0$ is a constant parameter reflecting the sensitivity of the quantity with respect to the price. The instantaneous profit on the established product market equals

$$\pi_0 = (\xi_0 - \alpha K_0)K_0. \tag{1}$$

As the firm produces up to capacity and therefore, the variable costs are constant, we simplify notation by omitting these costs.

The firm has an option to innovate, i.e. to adopt a new technology by which it can produce a new innovative product. To do so the firm has to incur an investment cost. We consider that this cost is proportional to the capacity level of the new product, K_1, specifically the cost is equal to δK_1, with $\delta > 0$. For the new product we also assume that the firm produces up to capacity.

Similar to Farzin et al. [1998] and Huisman [2001], the state of the technology is given by a compound Poisson process, $\boldsymbol{\theta} = \{\theta_t : t \geq 0\}$. We may express θ_t as $\theta_t = \theta_0 + uN_t$, where θ_0 denotes the state of technology at the initial point in time, $u > 0$ is the jump size and $\{N_t, t \geq 0\}$ follows a homogeneous Poisson process with rate $\lambda > 0$. The later the firm adopts the higher quality the product has, so the higher the demand for this product will be.

We denote the time of adoption of a new technology by τ_1. When the firm invests in the new product, it

[1] This assumption is often referred to as the market clearance assumption (see, e.g., R. Deneckere [1997], Anand and Girotra [2007] and Goyal and Netessine [2007]). Always producing up to capacity arises because firms may find it difficult to produce below capacity due to fixed costs associated with, for example, labour, commitments to suppliers, and production ramp-up (Goyal and Netessine [2007]). Even when firms can keep some capacity idle, a temporary suspension of production is often costly due to, for instance, maintenance costs needed to avoid deterioration of the equipment. Therefore, in practice firms often reduce prices to keep production lines running (see Mackintosh [2003], Anand and Girotra [2007] and Goyal and Netessine [2007]). However, counterexamples to the assumption of producing up to capacity also exist. Hagspiel et al. [2016b] showed that allowing the firm to produce below capacity leads to larger capacity investment while the effect on timing shows a tradeoff: on the one hand the firm likes to invest earlier as the project is more valuable due to this volume flexibility, but on the other hand the firm has an incentive to invest later because investing in a larger capacity is more costly.

has two options. It can decide to either produce both products, or to immediately replace the old product by the innovative one. In case the firm decides to replace the old product by the new one, the price of the new product satisfies:

$$p_1^R(X_t, \theta_{\tau_1}) = (\theta_{\tau_1} - \alpha K_1)X_t, \quad t \geq \tau_1.$$

The way inverse demand shifts with X follows the real option literature. It started with Dixit and Pindyck [1994], who consider $P = XD(Q)$ with $D(Q)$ unspecified. This variant, but then with Q being linear in $D(Q)$ has been adopted in, e.g., Huisman and Kort [2015] and Hagspiel et al. [2016b]. The demand curve having a parallel shift, which would be, for instance, $p_1^R = X - \alpha K$, would require a completely new analysis (for an application see Hagspiel et al. [2016a]).

If the new product is produced together with the established one, the demand system for the two products is given as follows[2]:

$$p_0^A(X_t, \theta_{\tau_1}) = \xi_0 - \alpha K_0 - \beta K_1 X_t, \qquad t \geq \tau_1,$$

$$p_1^A(X_t, \theta_{\tau_1}) = (\theta_{\tau_1} - \alpha K_1 - \beta K_0)X_t, \qquad t \geq \tau_1.$$

The new product is horizontally differentiated from the old one, where $\beta > 0$ represents the horizontal differentiation parameter. We assume β to be positive to reflect that the two products are substitutes. The upper bound of β is given by α ($\beta < \alpha$) meaning that it can never be the case that the quantity of the other product has a larger effect on the product price than the quantity of the product itself. Besides β, the mixeed terms in the demand system are also linearly dependent on X. There are two reasons for this. The first reason is that the mixed terms in the demand system should be connected such that they can be derived from a specific utility function (see footnote 2). Second, the economic argument is that, since the products are strategic substitutes, when demand for the innovative product goes up, demand for the old product goes down and vice versa.

[2]The demand system can be derived from the following utility function

$$U = \xi_0 K_0 - \frac{1}{2}\alpha K_0^2 - \beta K_0 K_1 X + \theta_{\tau_1} K_1 X - \frac{1}{2}\alpha K_1^2 X - p_0 K_0 - p_1 K_1.$$

The instantaneous profit function for the case that only the new product is produced is then equal to

$$\pi_1^R(X_t, \theta_{\tau_1}) = (\theta_{\tau_1} - \alpha K_1) X_t K_1, \quad t \geq \tau_1$$

while, in case that both products are produced, it is given by

$$\pi_1^A(X_t, \theta_{\tau_1}) = (\xi_0 - \alpha K_0 - \beta K_1 X_t) K_0 + (\theta_{\tau_1} - \alpha K_1 - \beta K_0) X_t K_1, \quad t \geq \tau_1.$$

Regarding the demand we will address the following two cases:

i) The demand of the new product is known beforehand, and deterministic. Therefore, we have $X_t = x$ for all $t \geq \tau_1$. We will refer to this case as the *benchmark case.*

ii) The demand of the new product is not known beforehand. It depends on, $\boldsymbol{X} = \{X_t, t \geq \tau_1\}$. Specifically, $\boldsymbol{X}$ follows a GBM with drift μ and volatility $\sigma > 0$, with $r - \mu > 0$ [3], where $r > 0$ is the (constant) interest rate. Typically for these new products the market is expected to be growing. Therefore, we assume that the drift is non-negative, i.e. $\mu \geq 0$. In order to make sure that the price of the old market stays always positive, we need to impose the additional assumption $r + \mu > \sigma^2$. We let x denote its initial value, i.e., x is the value of the demand at the time that the investment in the new technology takes place, meaning that $X_{\tau_1} = x$. This case will be referred to as the *stochastic case.*

Finally, we note that for the benchmark case, upon investment in the new technology the firm either replaces the old product right away or produces both products, forever, due to the assumption that the demand is fixed. In the stochastic case, the demand may fluctuate. Therefore, it can happen that the firm first adds the second product to the initial one, and only latter abolishes the original one. Therefore, the optimization problems have to be treated differently, depending on the case that we are considering.

In the next section we solve the optimal stopping time problems for each one of the considered cases.

3 Optimization Problem

In this section we derive the optimal decision regarding the following times: i) when to invest in the new technology, i.e. determine τ_1 (for the benchmark and for the stochastic cases); and ii) when to stop producing

[3]This is a standard assumption to ensure that the optimal investment time is finite.

the first product, which is denoted by τ_2, with $\tau_2 \geq \tau_1$ (only relevant for the stochastic case).

In general, the optimal stopping problem is defined as follows:

$$\begin{aligned} V_x(\theta) &= \sup_{\tau_1} \mathbb{E}\left[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \left\{ \sup_{\tau_2 : \tau_2 \geq \tau_1} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \pi_1^A(X_s, \theta_{\tau_1}) e^{-rs} ds - \delta K_1 e^{-r\tau_1} \right.\right.\right. \\ &\quad \left.\left.\left. + \left\{\int_{\tau_2}^{+\infty} \pi_1^R(X_s, \theta_{\tau_1}) e^{-rs} ds\right\} \chi_{\{\tau_2 < +\infty\}} \right| X_{\tau_1} = x\right]\right\} \chi_{\{\tau_1 < +\infty\}} \left| \theta_0 = \theta \right], \end{aligned} \tag{2}$$

for $\theta, x \in \mathbb{R}^+$, where $\chi_{\{A\}}$ represents the indicator function of set A. In the benchmark case the process $\boldsymbol{X}$ is constant and equal to x, whereas in the stochastic case it is a GBM with initial value x. Note that we have indexed the value function by x as it explicitly depends on this value, which is fixed exogenously. This will play an important role in the sequel, in particular, in the stochastic case.

3.1 Benchmark case

In the benchmark case, substituting X_s by x in (2) and doing simple transformations we get

$$\begin{aligned} V_x(\theta) &= \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[\int_0^{\tau_1} \pi_0 e^{-rs} ds + \left\{\int_{\tau_1}^{+\infty} \pi_1^A(x, \theta_{\tau_1}) e^{-rs} ds - \delta K_1 e^{-r\tau_1} \right.\right. \\ &\quad \left.\left. + \sup_{\tau_2 : \tau_2 \geq \tau_1} \left\{\int_{\tau_2}^{+\infty} \left[\pi_1^R(x, \theta_{\tau_1}) - \pi_1^A(x, \theta_{\tau_1})\right] e^{-rs} ds\right\} \chi_{\{\tau_2 < +\infty\}}\right\} \chi_{\{\tau_1 < +\infty\}}\right], \end{aligned}$$

where, in order to ease the notation, we denote the conditional expectation $\mathbb{E}[...|\theta_0 = \theta]$ by $\mathbb{E}^{\theta_0=\theta}[...]$. Therefore, the decision between replacing the old product by the new one, and producing both, depends only on the relationship between the revenues for each case. Moreover, the firm either decides to produce both products forever ($\tau_2 = +\infty$) or to replace the old product by the new one immediately ($\tau_2 = \tau_1$). Indeed, upon investment (at time τ_1) the firm should replace the old product by the new one if and only if

$$\pi_1^R(x, \theta_{\tau_1}) > \pi_1^A(x, \theta_{\tau_1}) \Leftrightarrow x > \frac{\pi_0}{2\beta K_0 K_1} \equiv x_B^*. \tag{3}$$

Consequently, the optimal stopping problem may be written as follows

$$\begin{aligned} V_x(\theta) &= \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[\int_0^{\tau_1} \pi_0 e^{-rs} ds \right. \\ &\quad \left. + \left\{\int_{\tau_1}^{+\infty} \left[\pi_1^A(x, \theta_{\tau_1}) \chi_{\{0 < x < x_B^*\}} + \pi_1^R(x, \theta_{\tau_1}) \chi_{\{x \geq x_B^*\}}\right] e^{-rs} ds - \delta K_1 e^{-r\tau_1}\right\} \chi_{\{\tau_1 < +\infty\}}\right]. \end{aligned}$$

which can be rewritten as follows

$$V_x(\theta) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1} \rho_x^B(\theta_{\tau_1})\right], \tag{4}$$

where

$$\rho_x^B(\theta) = \frac{(\theta - \alpha K_1)K_1 x}{r} - \delta K_1 - \frac{2\beta K_0 K_1 x}{r}\chi_{\{0<x<x_B^*\}} - \frac{\pi_0}{r}\chi_{\{x\geq x_B^*\}}. \tag{5}$$

3.2 Stochastic case

In the stochastic case the firm needs to decide about when to invest in the new market, and also for how long to produce both products. We remark that in this case, it can be optimal for the company to produce both products for a certain period - during (τ_1, τ_2) - and then abandon the first product at τ_2 and thereupon, only produce the new product. From simple manipulations and applying a change of variable in (2), we obtain

$$\begin{aligned} V_x(\theta) \quad &= \quad \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\left\{\sup_{\tau_2} \mathbb{E}^{X_{\tau_1}=x}\left[\int_0^{\tau_2-\tau_1}\left(\pi_1^A(X_{\tau_1+s}, \theta_{\tau_1}) - \pi_0\right)e^{-rs}ds \; - \delta K_1\right.\right.\right. \\ &\left.\left.\left. + \left\{\int_{\tau_2-\tau_1}^{+\infty}\left(\pi_1^R(X_{\tau_1+s}, \theta_{\tau_1}) - \pi_0\right)e^{-rs}ds\right\}\chi_{\{\tau_2<+\infty\}}\right]\right\}\chi_{\{\tau_1<+\infty\}}\right], \end{aligned}$$

where, similarly to θ, we denote the conditional expectation $\mathbb{E}\left[...|X_{\tau_1} = x\right]$ by $\mathbb{E}^{X_{\tau_1}=x}\left[...\right]$. Denoting τ as the time period in which the firm is producing both products (i.e. $\tau = \tau_2 - \tau_1$), we, therewith can rewrite the value function V as

$$\begin{aligned} V_x(\theta) \quad &= \quad \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\left\{\mathbb{E}^{X_{\tau_1}=x}\left[\int_0^{+\infty}\left(\pi_1^A(X_{\tau_1+s}, \theta_{\tau_1}) - \pi_0\right)e^{-rs}ds\right] - \delta K_1\right.\right. \\ &\left.\left. + \sup_{\tau} \mathbb{E}^{X_{\tau_1}=x}\left[\left\{\int_{\tau}^{+\infty}\left[\pi_1^R(X_{\tau_1+s}, \theta_{\tau_1}) - \pi_1^A(X_{\tau_1+s}, \theta_{\tau_1})\right]e^{-rs}ds\right\}\chi_{\{\tau<+\infty\}}\right]\right\}\chi_{\{\tau_1<+\infty\}}\right]. \end{aligned} \tag{6}$$

We now treat the two integrals separately. First, we derive

$$\mathbb{E}^{X_{\tau_1}=x}\left[\int_0^{+\infty}\left(\pi_1^A(X_{\tau_1+s}, \theta_{\tau_1}) - \pi_0\right)e^{-rs}ds\right] = \frac{\left(\theta_{\tau_1} - \alpha K_1 - 2\beta K_0\right)K_1 x}{r-\mu}, \tag{7}$$

using Fubini's theorem (see Hildebrandt [1963]) and the fact that the GBM has stationary increments. The integral convergence is guaranteed by the initial assumption $r > \mu$. Regarding the second integral, one note in view of the strong Markov property of the GBM (see Karlin [2014]) that $\{(X_t|X_{\tau_1} = x), t \geq \tau_1\} \stackrel{d}{=} \{(X_t|X_0 = x), t \geq 0\}$, and by Fubini's theorem, it follows that

$$\mathbb{E}^{X_{\tau_1}=x}\left[\int_{\tau}^{+\infty}\left[\pi_1^R(X_{\tau_1+s}, \theta_{\tau_1}) - \pi_1^A(X_{\tau_1+s}, \theta_{\tau_1})\right]e^{-rs}ds\right] = \mathbb{E}^{X_0=x}\left[e^{-r\tau}\left(\frac{2\beta K_0 K_1 X_\tau}{r-\mu} - \frac{\pi_0}{r}\right)\right]. \tag{8}$$

Plugging (7) and (8) into (6) we arrive to the following expression for the value of the firm

$$V_x(\theta) \quad = \quad \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\rho_x^S(\theta_{\tau_1})\chi_{\{\tau_1<+\infty\}}\right], \tag{9}$$

with

$$\rho_x^S(\theta) = F(x) + \frac{\left[\left(\theta - \alpha K_1 - 2\beta K_0\right)x - \epsilon\right]K_1}{r-\mu},$$

where

$$\epsilon = \delta(r-\mu), \tag{10}$$

and

$$F(x) = \sup_{\tau} \mathbb{E}^{X_0=x}\left[e^{-r\tau} g(X_\tau)\chi_{\{\tau<+\infty\}}\right] \quad \text{and} \quad g(x) = \frac{2\beta K_0 K_1}{r-\mu}x - \frac{\pi_0}{r}. \tag{11}$$

In fact, in the stochastic case, the optimization problem (2) can be seen as two optimization problems that need to be solved: one related with the optimal investment time in the new product (τ_1), and the other related with the time, from which on the firm produces only the innovative product ($\tau_2 = \tau_1 + \tau$). Note that F, defined in (11), is the value function for a standard investment problem, for which the solution is given in the following proposition.

Proposition 1 *The solution of the problem presented in* (11) *is given by*

$$F(x) = \begin{cases} a x^{d_1} & 0 < x < x_S^\star, \\ \frac{2\beta K_0 K_1}{r-\mu}x - \frac{\pi_0}{r} & x \geq x_S^\star, \end{cases}$$

for all $x > 0$, where

$$x_S^\star = \frac{K_b}{K_1}, \tag{12}$$

$$a = \frac{\pi_0}{r(d_1-1)} x_S^{\star\,-d_1} = \left[\frac{2\beta K_0 K_1}{d_1(r-\mu)}\right]^{d_1}\left[\frac{\pi_0}{r(d_1-1)}\right]^{1-d_1}, \tag{13}$$

with

$$K_b = \frac{d_1}{2(d_1-1)}\frac{(r-\mu)\pi_0}{r\beta K_0} \quad \textit{and} \quad d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{14}$$

Proof of Proposition 1 See Appendix 6.1.1 for the proof.

Using the expression derived for F in Proposition 1, we can rewrite

$$\rho_x^S(\theta) = \frac{(\theta - \alpha K_1)K_1 x}{r-\mu} - \delta K_1 + \left(a x^{d_1} - \frac{2\beta K_0 K_1 x}{r-\mu}\right)\chi_{\{0<x<x_S^*\}} - \frac{\pi_0}{r}\chi_{\{x \geq x_S^*\}}. \tag{15}$$

3.3 General result

Next we present a general Theorem that is the basis to prove Proposition 2. We highlight that this theorem is general enough to be applied in other stopping time problems that involve decisions regarding technology innovations driven by compounded Poisson processes, as the problems addressed in Huisman and Kort [2004] or Hagspiel et al. [2016a].

Theorem 1 *Let us consider the optimal stopping problem* $G(\theta) = \sup_\tau \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau} g(\theta_\tau)\chi_{\{\tau<+\infty\}}\right]$*, where* $\boldsymbol{\theta} = \{\theta_t : t > 0\}$ *is a compound Poisson process, with rate* $\lambda > 0$ *and jump size* $u > 0$*, and* g *is a continuous function. Let us also assume that*

$$\exists\,!\,\theta^\star > 0 \;:\; h(\theta) > 0 \Leftrightarrow \theta > \theta^\star, \tag{16}$$

where $h(\theta) = (r+\lambda)g(\theta) - \lambda g(\theta+u)$*. Then, the solution of the problem is given by*

$$G(\theta) = \begin{cases} \left(\frac{\lambda}{\lambda+r}\right)^{n(\theta)} g(\theta + n(\theta)u) & 0 < \theta < \theta^\star, \\ \\ g(\theta) & \theta \geq \theta^\star, \end{cases} \tag{17}$$

with $n(\theta) = \left\lceil \frac{\theta^\star - \theta}{u} \right\rceil$*, where, for* $k \geq 0$*,* $\lceil k \rceil = \min\{n \in \mathbb{N} : n \geq k\}$*.*

Proof of Theorem 1 See Appendix 6.1.2 for the proof.

For the solution of the problems (4) and (9), one can use the result of Theorem 1, replacing the function g by

$$\rho_x^B(\theta) = \frac{(\theta - \alpha K_1)K_1 x}{r} - \delta K_1 - \frac{2\beta K_1 x K_0}{r}\chi_{\{0<x<x_B^*\}} - \frac{\pi_0}{r}\chi_{\{x>x_B^*\}}.$$

for the benchmark case, and

$$\rho_x^S(\theta) = \frac{(\theta - \alpha K_1)K_1 x}{r-\mu} - \delta K_1 + \left(a x^{d_1} - \frac{2\beta K_0 K_1 x}{r-\mu}\right)\chi_{\{0<x<x_S^*\}} - \frac{\pi_0}{r}\chi_{\{x\geq x_S^*\}}.$$

for the stochastic case, respectively. Therewith, we can finally present the solution of the optimization problem (2) including the exercise boundaries. This is presented in Proposition 2, for the benchmark and for the stochastic case, respectively.

Proposition 2 *The solution of the optimal stopping problem* (2) *is given by*

$$V_x(\theta) = \frac{\pi_0}{r} + \begin{cases} \left(\frac{\lambda}{\lambda+r}\right)^{n_x(\theta)} \rho_x(\theta + n_x(\theta)\ u) & 0 < \theta < \theta^*(x) \\ \rho_x(\theta) & \theta \geq \theta^\star(x) \end{cases} \tag{18}$$

for all $x, \theta \in \mathbb{R}^+$ *, where* $n_x(\theta) = \left\lceil \frac{\theta^\star(x)-\theta}{u} \right\rceil$ *and*

$$\theta^\star(x) \quad = \quad \upsilon^A(x)\chi_{\{0<x<x^\star\}} + \upsilon^R(x)\chi_{\{x \geq x^\star\}}. \tag{19}$$

Moreover, for each case we have the following definitions [4]*.*

- *Benchmark case:* x^* *is given in* (3)*,* ρ_x *is defined at* (5)*, and*

$$\upsilon^A(x) = \upsilon^A_B(x) := \frac{r\delta}{x} + \frac{\lambda u}{r} + \alpha K_1 + 2\beta K_0 \quad \textit{and} \quad \upsilon^R(x) = \upsilon^R_B(x) := \frac{r\delta}{x} + \frac{\lambda u}{r} + \alpha K_1 + \frac{\pi_0}{K_1 x}.$$

- *Stochastic case:* x^* *is given in* (12)*,* ρ_x *is defined at* (15)*, and*

$$\upsilon^A(x) = \upsilon^A_S(x) := \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + 2\beta K_0 - \frac{a(r-\mu)x^{d_1-1}}{K_1} \quad \textit{and} \quad \upsilon^R(x) = \upsilon^R_S(x) := \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + \frac{(r-\mu)\pi_0}{rK_1 x},$$

where π_0*,* ϵ*,* a *and* d_1 *are defined in* (1)*,* (10)*,* (13) *and* (14)*, respectively.*

Proof of Proposition 2 See Appendix 6.1.3 for the proof.

Proposition 2 presents the optimal value function, as well as the boundary curve, $\theta^\star(x)$. For technology levels smaller than this boundary (i.e. $0 < \theta < \theta^\star(x)$), it is optimal to wait with adoption of a new technology. At the moment this boundary curve is passed, it is optimal to adopt the current technology level. Figure 1 presents an illustration of this boundary curve in the (θ, x) – plane (bold line). For values to the lower left of the threshold curve $\theta^\star(x)$, it is optimal to wait with adoption and continue producing only the established product. As soon as the technology process passes the threshold curve for a given value of x, it is optimal to invest and therewith, introduce the innovative product to the market.

The optimal product portfolio decision can be interpreted as follows: at the optimal time τ_1 the firm invests in a certain technology level θ_{τ_1}. Depending on the value of the initial price intercept x, and given

[4]As we have been doing before, in order to ease the notation, we use the subscripts B and S to denote, respectively, the benchmark and the stochastic cases.

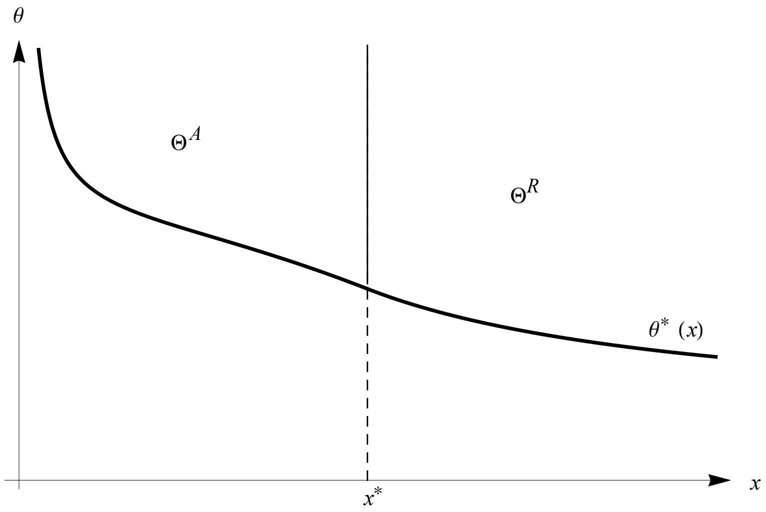


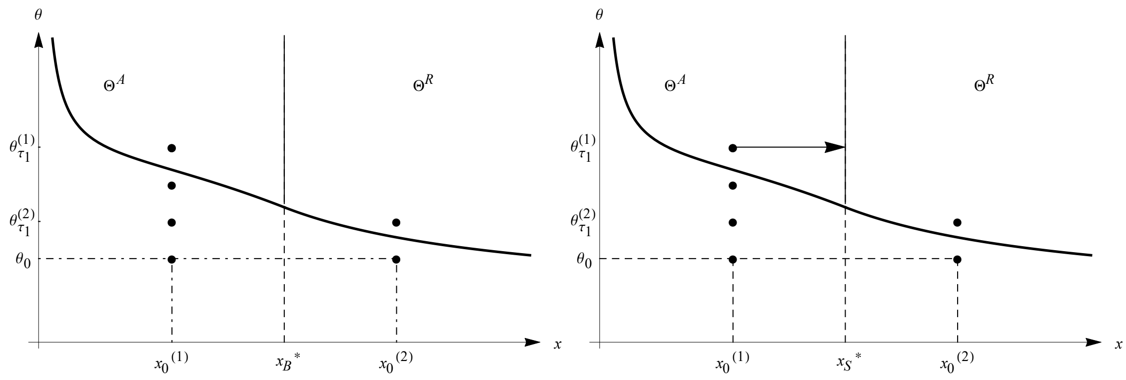
Figure 1: Plan division.

the technology level adopted (θ_{τ_1}), the firm will produce both products (for a certain time period in the stochastic case) after investment, or replace the old one by the innovative one immediately. We also define the following sets

$$\begin{aligned}\Theta^A &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x < x^\star \wedge \theta > \upsilon^A(x)\right\} \\ \Theta^R &= \left\{(x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^+ : x \geq x^\star \ \wedge \ \theta > \upsilon^R(x)\right\},\end{aligned}$$

where in the benchmark case $x^* = x_B^*$, $\upsilon^A = \upsilon_B^A$, and $\upsilon^R = \upsilon_B^R$, and in the stochastic case $x^* = x_S^*$, $\upsilon^A = \upsilon_S^A$ and $\upsilon^R = \upsilon_S^R$. The set Θ^A represents the region where the firm produces both products, whereas Θ^R is the region where it is optimal to replace the original product by the innovative. See Figure 1 for a representation of these two regions.

In Figure 2 we illustrate the difference of the two cases regarding the technology adoption and product portfolio decisions of the firm. Figure 2a refers to the benchmark case, should be read as follows: given that the current technology level is equal to θ_0, and the initial demand intercept is equal to x_0, it is optimal to wait for new technology improvements. When the level of the technology hits or exceeds the threshold $\theta^\star(x_0)$ (illustrated by the bold curve), the firm undertakes the investment. If the price intercept is such that $0 < x_0 < x_B^*$, then upon investment, the firm produces both products. In case $x_0 \geq x_B^*$, the firm replaces the old product by the new one upon investment.



(a) Benchmark case. (b) Stochastic case.

Figure 2: Possible movements of the bivariate process (x, θ).

For the stochastic case, we refer to the corresponding Figure 2b. It illustrates that, given that the firm is currently producing according to the technology level θ_0 and the initial value of the process $\boldsymbol{X}$ is such that $0 < x_0 < x_S^\star$, it is optimal for the firm to wait for new technology improvements. When the level of the technology hits or exceeds the threshold $\theta^\star(x_0)$, the firm undertakes the investment. From time τ_1 on, the process $\boldsymbol{X}$, with initial value x_0, evolves, which is represented by a horizontal movement in Figure 2b. As soon as the demand intercept process $\boldsymbol{X}$ hits the level $x_S^\star$ for the first time, the firm abandons the established product and produces only the innovative one from then on.

Next we study the relative position of the investment thresholds for the benchmark and the stochastic cases. First we compare the add/replace thresholds (3) and (12), in the two cases. The result is presented in the following proposition.

Proposition 3 *The replace decision in the stochastic case occurs for higher levels of the demand intercept than in the benchmark case, i.e.,* $x_S^* > x_B^*$.

Proof of Proposition 3 See Appendix 6.1.4 for the proof.

In view of this result, it follows that for values of x such that $x_B^* < x < x_S^*$, the decision in the benchmark case is to replace the old product by the new one, whereas in the stochastic case the new product is added to the established one. However, unlike in the benchmark case, in the stochastic case X is fluctuating over time, implying that there is a positive probability that the firm will still abolish producing the established product in the future, which happens once X reaches the level x_S^*.

Next we compare the threshold curves for the two cases.

Proposition 4 *For the same value of the current demand intercept x, the firm invests earlier in the stochastic case than in the benchmark case.*

Proof of Proposition 4 See Appendix 6.1.5 for the proof.

The intuition is that it is more attractive to invest in the stochastic case, because, first, X is expected to increase because of the positive trend parameter μ, which means that expected demand is increasing over time. Second, when it invests in the add domain, it acquires the option to replace, and the value of this option is especially large when uncertainty is high.

4 Comparative statics

In this section we study the behaviour of the add/replace boundary and the decision threshold with the different parameters and analyze the difference regarding, whether the demand for the innovative product is stochastic or not.

We start by analyzing how the add/replace boundaries, $x_B^\star$ and $x_S^\star$, are affected by a change in several parameters.

Proposition 5 *The add/replace boundary for the benchmark case, $x_B^\star$, increases with ξ_0, decreases for α, β, K_0 and K_1, and it is constant with r, δ, λ and u. For the stochastic case the add/replace boundary, $x_S^\star$, increases with ξ_0 and σ, decreases with $\alpha, \beta, K_0, K_1, \mu$ and r, and it is constant with δ, λ and u.*

Proof of Proposition 5 See Appendix 6.2.1 for the proof.

Proposition 5 shows that accounting for stochasticity of the new product demand, the add/replace boundary is decreasing in the discount rate, while it is constant in r when we assume that the demand for the new product is deterministic. The reason for that is related to the fact that by choosing to add the new product to the product portfolio, the firm gets the option to replace eventually in the stochastic case. In the benchmark case, however, the firm would keep on producing both products forever. The firm has a higher incentive to add the innovative product to the product portfolio if the discount rate r is small in the stochastic case. In that case the option to replace eventually is of high value and therefore, the firm wants to keep this option alive. If the interest rate increases, however, the firm is more myopic and values the replace option less. Therefore, it is more inclined to replace the old product immediately by the innovative one.

The add/replace boundary naturally only depends on μ and σ for the stochastic case. Proposition 5 shows that the firm is more inclined to first add the innovative product, i.e. $x_S^\star$ is larger, if the demand volatility for the new product σ is large. Regarding the drift the opposite holds. If μ increases the innovative product market becomes more attractive and therefore, the firm has more incentive to increase the instantaneous profit of the innovative product by getting rid of the cannibalization effect. If demand uncertainty, however, goes up, it is known from real options theory that it is optimal to delay irreversible decision. In this case it means that the firm wants to delay leaving the old market. This implies that the add region gets larger and the eventual switch to replace will occur for a larger value of x (i.e. later). Therefore, $x_S^\star$ increases with σ.

The comparative statics of the other parameters are not affected by introducing stochasticity for the innovative product demand. We show that the add/replace boundary is decreasing in α due to the fact that the old market becomes less attractive with a higher α, while the cannibalization effect stays the same. The firm looses less revenue on the old market and therewith, has more incentive to switch to only producing the innovative product. Regarding to the cannibalization parameter, β, we notice that the stronger the cannibalization effect the less attractive it is for the firm to produce both products, and therefore, replacing the old product becomes more attractive. In all previous cases replace gets more attractive relative to add.

Concerning the capacities, the add/replace boundary is also decreasing with both of them. The higher the capacity of the innovative product, K_1, the larger the cannibalization effect. This hurts the profit of add so that replace becomes more attractive. Relatively to the capacity of the old product, K_0, three effects can be distinguished. Due to an increase of K_0, the $x_S^\star$ decreases because the output price on the old market becomes lower and because of the increased cannibalization effect. On the other hand, a higher K_0 leads to a larger quantity on the old market and this has a positive effect on the $x_S^\star$. It turns out that the latter effect cancels against the cannibalization effect. Hence, $x^\star$ (for both the benchmark and the stochastic case) decreases because the market price of the old product becomes lower.

Furthermore, we show that the add/replace boundary is increasing in the demand intercept for the old product, ξ_0. The higher the demand intercept for the old product, the higher the value of the old product and therefore, the firm is more hesitant to replace it so that $x_S^\star$ gets larger.

In the following propositions we present the behaviour of the investment threshold with the relevant parameters, for the benchmark (Proposition 6) and the stochastic case (Proposition 7), respectively. We note that the comparative statics for the investment threshold involves to compare curves rather than points.

Proposition 6 *For the benchmark case, the investment threshold, $\theta_B^\star$, increases with δ, λ and u, and decreases with x; increases with β in the add region, and stays constant in the replace region; increases with K_0, K_1 and α in the add region, and does not have a monotonic behaviour in the replace region; stays constant with ξ_0 in the add region, and increases in the replace region; and does not have a monotonic behaviour with r.*

Proof of Proposition 6 See Appendix 6.2.2 for the proof.

Proposition 7 *For the stochastic case, the investment threshold, $\theta_S^\star$, increases with δ, λ, u, and ξ_0, and decreases with μ and x; decreases/increases with σ/β in the add region, and stays constant in the replace region; and does not have a monotonic behaviour with K_0, K_1, α and r.*

Proof of Proposition 7 See Appendix 6.2.3 for the proof.

We now interpret the results of Proposition 7 (stochastic case). The results of Proposition 6 are mainly the same in a qualitative sense, but there where they are different we explain why.

When accounting for demand uncertainty in the innovative product market, we show that the investment threshold curve is decreasing in demand volatility of the innovative product market given that the firm stays in the old market for a given time upon investment. This at first sight counterintuitive result stems from the fact that, upon adding the new product to the product portfolio, the firm gains the option to eventually replace the old product. The value of this option is increasing in volatility, which in turn increases the value of investment and therefore, the firm is more eager to invest early. If the firm decides to replace the old product immediately, the demand volatility of the innovative product market does not have an effect on the innovation decision.

The investment threshold is also decreasing in the drift, μ. The larger the growth of the innovative product market, the more attractive the market is and therefore, the firm invests sooner. The same holds for the initial value x of the demand intercept at the time the investment in the new technology is made, as a higher x makes the innovative product market more attractive. This in turn means that, the smaller the initial demand for the innovative product market, the better the quality of the innovative product needs to be for the product innovation to be optimal.

As expected the parameter β will only affect the investment threshold when the firm will produce both the old as well as the innovative product right after the investment. Since this parameter enhances the cannibalization effect, a larger β makes investment in that case less attractive.

Moving to the parameters with which the investment threshold is always increasing. Regarding the unit investment cost δ, we notice that the higher the costs for a given capacity K_1 the higher the technology level needs to be for the firm to justify investment. With regard to λ and u, $\theta^\star$ increases as well because it pays more for the firm to wait for the next technology jump if this is effected to arrive sooner and/or when this jump is larger in size. These results are robust for the benchmark and stochastic model. Furthermore, the investment threshold is increasing with ξ_0, as the old market is more profitable for a higher ξ_0 and therefore, the firm waits for a higher technology level to justify investment.

The investment threshold does not have a monotonic behaviour with the remaining parameters. In some cases there are several effects driven the movements, which makes the interpretation very complex or not understandable. Hence, in this case we omit an explanation for the effect of the interest rate, r.

In general the firm invests later for a higher α because it has to wait for a higher technology level to justify the capacity K_1. However, three other effects can be identified that in total could result in investing earlier when α goes up. The first effect is what we refer to as the *option effect.* It explains that the option to replace after having added the innovative product in the first place, is smaller for higher α. The second effect has to do with the revenue before the firm invests. The higher the α the smaller the revenue on the old market on which the firm solely produces before the investment. We refer to this as the *opportunity* *cost effect*. In Proposition 5 we have established that the add/replace boundary $x^\star$ decreases with α. So if α goes up it could happen that the firm changes from an add to a replace strategy. This implies that no cannibalization takes place anymore and therefore, the firm will invest earlier. This is the third effect which we call the *cannibalization effect*.

Finally, the investment threshold is non-monotonic, in both the capacity of the old and the capacity of the new product. The investment threshold in the benchmark case is decreasing in K_1 for low values and increasing for high values of K_1, following a typical U − shape with the smallest threshold value for the optimal capacity level, at which the value of investment in the innovative product is the highest, and therefore, investment optimal earliest. With respect to K_0 an inverse U − shape can be observed. The threshold is increasing for low values of K_0 and decreasing for high values. At the optimal capacity level for the old product, investment in the innovative product is the least appealing.

In the benchmark case we have a similar behavior for replace, but for add the threshold is monotonically increasing in both K_0 and K_1. The latter result holds, because the deterministic demand for the new product implies that there is no uncertainty regarding the value of the option to replace, which reduces this option value considerably. Therefore, the cannibalization effect is the dominant factor. This cannibalization effect increases in both K_0 and K_1, making investment less attractive. Hence, the threshold is monotonically increasing in both K_0 and K_1. In addition, this effect regarding K_1 is reinforced by the fact that for a larger K_1 the investment costs are also larger.

4.1 Numerical Illustration

In the following, we compute the thresholds for the benchmark and for the stochastic case, using the base-case parameters presented in Table 1, which are similar to the ones used in Hagspiel et al. [2016b] and Huisman [2013].

The corresponding decision thresholds are presented in Table 2. For these values, the thresholds for the benchmark and for the stochastic case are not very different. In both cases we have the same decision regarding keeping the old product in the market just after the investment in the new product.

In Table 3 we illustrate the behavior of the thresholds when we change the capacity before investment

r	risk-free rate	0.05
μ	drift	0.02
σ	volatility	0.1
K_0	initial capacity	200
K_1	capacity after investment	300
λ	Poisson intensity	0.1
u	jump size	0.5
θ_0	initial technology level	1
x	constant price before investment	1
ξ_0	maximum willingness	30
α	sensitivity of the quantity w.r.t. price	0.1
β	horizontal differentiation	0.05
δ	investment cost	100

Table 1: Base–case parameters used to calculate the thresholds

	x^*	$\theta^\star$	decision
Benchmark case	0.33	42.7	add
Stochastic case	0.40	38	add

Table 2: Thresholds for the base–case parameters

(K_0). The main conclusion is that when the investment in the new market is considerable larger than in the old market (i.e, when K_1 is considerably larger than K_0), the decision in the benchmark case is to replace the old product by the new one, whereas in the stochastic case the firm will produce both products during some period. When K_0 increases, both situations lead to the replace decision. We remark also that the investment threshold is not monotonic with increasing K_0, for both cases (benchmark and stochastic), which is according to our findings of Proposition 6.

Finally, we show how the thresholds change with the capacity after investment (K_1) (see Table 4). The

	x_B^*	θ_B^*	decision (benchmark)	x_S^*	θ_S^*	decision (stochastic)
$K_0 = 25$	0.92	38.3	replace	1.1	35.4	add
$K_0 = 50$	0.83	40.2	replace	1.0	36.5	add
$K_0 = 100$	0.67	42.7	replace	0.8	38.0	replace
$K_0 = 200$	0.33	42.7	replace	0.4	38.0	replace
$K_0 = 220$	0.27	41.9	replace	0.32	37.5	replace

Table 3: Behavior of thresholds with K_0

results show that once K_1 is larger than K_0 (i.e., the firm is investing in more capacity) the firm replaces the old product by the new one, both in the benchmark and in the stochastic case.

	x_B^*	θ_B^*	decision (benchmark)	x_S^*	θ_S^*	decision (stochastic)
$K_1 = 50$	2	31	add	2.4	24.8	add
$K_1 = 100$	1	36	add	1.2	25.7	add
$K_1 = 200$	0.5	36	replace	0.6	30	replace
$K_1 = 300$	0.33	42.7	replace	0.4	38	replace

Table 4: Behavior of thresholds with K_1

5 Conclusion

This paper studies a setting where the firm, currently producing an established product, has the option to invest in a more innovative technology, in order to boost its profits by introducing a new, innovative product on the market. Moreover, the firm may decide whether to add the new product to its product portfolio, or to immediately replace the old product upon investment. In the earlier case it can decide to eventually stop the production of the old product at a later point in time.

We find that if the innovative product market is more volatile, the firm has more incentive to first add the innovative product to the product portfolio, and only replace the old product eventually. If the interest rate is larger, however, the firm is more inclined to replace the old product immediately.

Contrary to the standard investment problem, the investment threshold is not a single point but represented by a threshold curve. We find that the threshold curve is decreasing in the demand volatility of the innovative product market, and therefore, the firm innovates sooner, given that the firm will be active on the old market for at least some time. Furthermore, the threshold curve is decreasing in the initial demand of the innovative product, implying that the smaller the initial demand, the better the quality of the new product needs to be before it is optimal to innovate.

This paper is the first in a series of contributions that concentrates on when a firm should innovate and with how much. At the same time it also has to decide on whether to keep on producing the established product. The present paper focuses on the innovation timing decision and on the question what to do with the established product. The plan is that follow up papers will deal with the optimal determination of the capacity size for the established and the innovative product. It would be interesting to establish to what extent our present results will change when the firm can control the capacities.

6 Appendix

6.1 Proofs of the optimal stopping problems

6.1.1 Proof of Proposition 1

We want to solve the optimal stopping time (11), i.e.

$$F(x) = \sup_{\tau} \mathbb{E}^{X_0=x} \left[e^{-r\tau} g(X_\tau) \chi_{\{\tau<+\infty\}} \right], \tag{20}$$

where

$$g(x) = \frac{2\beta K_0 K_1}{r-\mu} x - \frac{\pi_0}{r}.$$

The corresponding Hamilton-Jacobi-Bellman (HJB, for short) equation for the optimization problem is given by

$$\min\{rF(x) - \mathcal{L}_X F(x), F(x) - g(x)\} = 0$$

where $\mathcal{L}_X$ is the infinitesimal generator of the process $\boldsymbol{X}$, i.e.

$$\mathcal{L}_X f(x) = \frac{\sigma^2}{2} x^2 f''(x) + \mu x f'(x).$$

By construction, in the stopping region we trivially have $F(x) = g(x)$. Moreover, the continuation region, hereby denoted by $\mathcal{C}_X$, must contain the following set (see Øksendal [2014] for details)

$$\mathcal{U}_X = \{u > 0 : rg(u) - \mathcal{L}_X g(u) < 0\} = \left(0, \frac{\pi_0}{2\beta K_0 K_1}\right)$$

which implies that $\mathcal{C}_X = (0, x_S^\star)$, where $x_S^\star$ still needs to be derived, such that $x_S^\star \geq \frac{\pi_0}{2\beta K_0 K_1}$.

In the continuation region the function F must satisfy the left hand side of the HJB equation. Let us define ζ as the solution of the equation $r\zeta(x) - \mathcal{L}_X \zeta(x) = 0$, that is

$$\frac{\sigma^2}{2} x^2 \zeta''(x) + \mu x \zeta'(x) - r\zeta(x) = 0. \tag{21}$$

The differential Equation (21) is a Cauchy-Euler equation, which means that the solution is given by

$$\zeta(x) = a x^{d_1} + b x^{d_2}, \tag{22}$$

where d_1 and d_2 are the positive and negative solutions, respectively, of the quadratic equation

$$\frac{\sigma^2}{2} d(d-1) + \mu d - r = 0, \tag{23}$$

given by

$$d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \qquad \text{and} \qquad d_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Given that $r > \mu$, it follows that $d_1 > 1$ and $d_2 < 0$.

We note that this optimization problem is in fact a special case of the case studied by Guerra et al. [2016]. The profit function g is a non-decreasing function of polynomial type, as considered by the referred authors. Then, one of the boundary conditions is that the solution for $x = 0$ needs to be zero, i.e. $\lim_{x \to 0^+} \zeta(x) = 0$. Therefore, we must have $b = 0$, and thus $\zeta(x) = a x^{d_1}$.

As the value function needs to be continuous and smooth in all its domain and, in particular, in $x_S^\star$, then it follows from the smooth pasting conditions, $F(x_S^\star) = g(x_S^\star)$ and $F'(x) = F'(x)|_{x = x_S^\star}$ (for more details see Øksendal [2014]), that $x_S^\star$ and a are as presented in Expressions (12) and (13), respectively. The remain of the proof is to check that $F(x) = \zeta(x)\chi_{\{0<x<x_S^\star\}} + g(x)\chi_{\{x \geq x_S^\star\}}$ is indeed the solution of the HJB equation. For that we need to prove that:

- in the stopping region $rg(x) - \mathcal{L}_X g(x) \geq 0$.

  As $rg(x) - \mathcal{L}_X g(x) = 2\beta K_0 K_1 x - \pi_0$, it will be positive if and only if $x > \frac{\pi_0}{2\beta K_0 K_1}$. The result holds in case $x_S^\star > \frac{\pi_0}{2\beta K_0 K_1}$, which, in view of the expression for $x_S^\star$, is equivalent to

  $$\frac{r-\mu}{r}\frac{d_1}{d_1-1} > 1 \Leftrightarrow r - \mu d_1 > 0.$$

  Recalling the d_1 definition, which comes from Equation (23), we have

  $$r - \mu d_1 = \frac{\sigma^2}{2} d_1(d_1 - 1),$$

  which is always positive. Thus $x_S^\star > \frac{\pi_0}{2\beta K_0 K_1}$ and therefore $rg(x) - \mathcal{L}_X g(x) \geq 0$ for $x \geq x_S^\star$.

- in the continuation region $\zeta(x) \geq g(x)$.

  By construction g is tangent with ζ at point $x_S^\star$. Moreover $g(0) = -\frac{\pi_0}{r} < 0$, $\zeta(0) = 0$ and ζ is a convex function (because $a > 0$ and $d_1 > 1$). Therefore, by Boyd and Vandenberghe [2004], ζ must be above all its tangents, and thus, in particular, is above g.

Therefore, we conclude that F is indeed the solution of the HJB equation, which ends the proof.

■

6.1.2 Proof of Theorem 1

We want to solve the problem $G(\theta) = \sup_\tau \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau} g(\theta_\tau)\chi_{\{\tau<+\infty\}}\right]$, for which the HJB equation is given by

$$\min\{rG(\theta) - \mathcal{L}_\theta G(\theta), G(\theta) - g(\theta)\} = 0,$$

with $\mathcal{L}_\theta$ being the infinitesimal generator of the compounded Poisson process $\boldsymbol{\theta}$, $\mathcal{L}_\theta\, l(\theta) = \lambda\left[l(\theta+u) - l(\theta)\right]$. In the continuation region we should have $G(\theta) = \frac{\lambda}{\lambda+r} G(\theta+u)$ and $G(x) \geq g(x)$. In the stopping region we should have $G(\theta) = g(\theta)$ and $rg(\theta) - \mathcal{L}_\theta g(\theta) \geq 0$. Considering the function $h(\theta) = (r+\lambda)g(\theta) - \lambda g(\theta+u)$, which is continuous, and taking into account the Condition (16), we realize that $h(\theta^\star) = 0 \Leftrightarrow \frac{\lambda}{\lambda+r} G(\theta^\star + u) = g(\theta^\star)$, i.e. $\theta^\star$ is exactly the threshold between the continuation and the stopping regions. Given Condition (16), we propose that the continuation region stands for lower values of θ and the stopping region stands for higher values of θ. Therefore, we may write

$$G(\theta) = \begin{cases} \frac{\lambda}{\lambda+r} G(\theta+u) & 0 < \theta < \theta^\star \\ \\ g(\theta) & \theta \geq \theta^\star \end{cases}$$

Using a backwards iterative reasoning, we can conclude that for θ such that $\theta^\star - nu \leq \theta < \theta^\star - (n-1)u$, with $n \in \mathbb{N}$ such that $\theta^\star - nu \geq 0$, we have

$$G(\theta) = \left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu).$$

This expression can be written in a more compact way as presented in (17).

To finish the proof, it remains to check $rg(\theta) - \mathcal{L}_\theta g(\theta) \geq 0$ when $\theta \geq \theta^\star$, and $G(\theta) \geq g(\theta)$ when $0 < \theta < \theta^\star$. The first part comes instantly from Condition (16). For the second part, we must prove that $\left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu) \geq g(\theta)$ for $\theta^\star - nu \leq \theta \leq \theta^\star - (n-1)u$, with $n \in \mathbb{N}$ such that $\theta^\star - nu \geq 0$. In order to prove it, we start realizing that

$$\left(\frac{\lambda}{\lambda+r}\right)^{n-1} g(\theta+(n-1)u) \leq \left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu) \Leftrightarrow h\left(\theta+(n-1)u\right) \leq 0 \Leftrightarrow \theta \leq \theta^\star - (n-1)u.$$

This imply that, for a specific $n \in \mathbb{N}$ and $\theta > 0$, such that $\theta \leq \theta^\star - (n-1)u < ... < \theta^\star - 2u < \theta^\star - u < \theta^\star$, we have

$$g(\theta) \leq \left(\frac{\lambda}{\lambda+r}\right) g(\theta+u) \leq \left(\frac{\lambda}{\lambda+r}\right)^2 g(\theta+2u) \leq ... \leq \left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu).$$

So, we proved that $\left(\frac{\lambda}{\lambda+r}\right)^n g(\theta+nu) \geq g(\theta)$ for $\theta^\star - nu \leq \theta \leq \theta^\star - (n-1)u$.

Therefore, we conclude that function G, given by (17), is indeed the solution of the HJB equation, which ends the proof.

■

6.1.3 Proof of Proposition 2

Here we provide the proof for the stochastic case, as the benchmark case is similar and simpler. For the stochastic case, we want to solve the problem (9), which is given by

$$V_x(\theta) = \frac{\pi_0}{r} + \sup_{\tau_1} \mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\rho_x^S\left(\theta_{\tau_1}\right)\chi_{\{\tau_1<+\infty\}}\right],$$

where

$$\rho_x^S(\theta) \quad = \quad \frac{[(\theta-\alpha K_1)\,x-\epsilon]\,K_1}{r-\mu}+\left(ax^{d_1}-\frac{2\beta K_0 K_1 x}{r-\mu}\right)\chi_{\{0<x<x_S^\star\}}-\frac{\pi_0}{r}\chi_{\{x\geq x_S^\star\}}$$

with $x_S^\star$, a and d_1 defined in (12), (13) and (14), respectively.

For each $x>0$, the optimal stopping problem $G_x(\theta)=\sup_{\tau_1}\mathbb{E}^{\theta_0=\theta}\left[e^{-r\tau_1}\rho_x^S(\theta_{\tau_1})\chi_{\{\tau_1<+\infty\}}\right]$ is of the same type as that presented in Theorem 1. Given that ρ_x^S is a continuous function, we only need to prove that Condition (16) holds, in order to have the solution.

Let us consider the function $h_x(\theta)=(r+\lambda)\rho_x^S(\theta)-\lambda\rho_x^S(\theta+u)$, which can be written as

$$h_x(\theta) \quad = \quad \frac{[[r\,(\theta-\alpha K_1)-\lambda u]\,x-r\epsilon]\,K_1}{r-\mu}+r\left(ax^{d_1}-\frac{2\beta K_0 K_1 x}{r-\mu}\right)\chi_{\{0<x<x_S^\star\}}-\pi_0\chi_{\{x\geq x_S^\star\}}.$$

Notice that, for a fixed x, h_x is an increasing linear function in θ, with zero at

$$\gamma_x=\frac{\epsilon}{x}+\frac{\lambda u}{r}+\alpha K_1+\left(2\beta K_0-\frac{a(r-\mu)}{K_1}x^{d_1-1}\right)\chi_{\{0<x<x_S^\star\}}+\frac{(r-\mu)\pi_0}{rK_1x}\chi_{\{x\geq x_S^\star\}}.$$

Obviously, if $x\geq x_S^\star$ we certainly have $\gamma_x>0$. If $0<x<x_S^\star$ then, in view of the definitions of a and $x_S^\star$, it follows that $2\beta K_0K_1-a(r-\mu)x^{d_1-1}>2\beta K_0K_1-a(r-\mu){x_S^\star}^{d_1-1}=\frac{2\beta K_0K_1(d_1-1)}{d_1}>0$. Therefore $\gamma_x>0$ for all $x>0$. Hence $\theta_x^\star=\gamma_x$ is the only zero of h_x, and $h_x(\theta)>0\Leftrightarrow\theta>\theta^\star$, which means that Condition (16) holds. By Theorem 1, we conclude that function V_x, given by (18), is indeed the solution of the optimal stopping problem (9).

■

6.1.4 Proof of Proposition 3

In the proof of Proposition 1 we already have shown that $x_S^\star>\frac{\pi_0}{2\beta K_0K_1}$, which is exactly what we want to prove here.

■

6.1.5 Proof of Proposition 4

We want to investigate the relative position of the threshold curves $\theta_B^\star$ and $\theta_S^\star$, where

$$\theta_B^\star(x)=\upsilon_B^A(x)\chi_{\{0<x<x_B^\star\}}+\upsilon_B^R(x)\chi_{\{x\geq x_B^\star\}}$$

and

$$\theta_S^\star(x) = \upsilon_S^A(x)\chi_{\{0<x<x_S^\star\}} + \upsilon_S^R(x)\chi_{\{x\geq x_S^\star\}}.$$

We start calculating the difference of the respective functions in the add and replace regions, namely,

$$\upsilon_B^A(x) - \upsilon_S^A(x) > 0 \Leftrightarrow x^{d_1} > -\frac{\mu\delta K_1}{a(r-\mu)}$$

and

$$\upsilon_B^R(x) - \upsilon_S^R(x) > 0 \Leftrightarrow \frac{\mu}{x}\left(1 + \frac{\pi_0}{rK_1}\right) > 0.$$

Given that we are assuming that $\mu > 0$, both previous inequalities hold for $x > 0$. Furthermore, both threshold curves are continuous and decrease with x, then we conclude that $\theta_B^\star(x) > \theta_S^\star(x)$, for all $x > 0$.

■

6.2 Proofs for comparative statics

In the comparative statics it is important to highlight the dependence of the functions in each parameter. For ease the notation, when we want to emphasize the dependency of one quantity (a, say) with one parameter (β, say) we simply write the dependency on that parameter, assuming the others constant ($a(\beta)$, say).

We start studying how the parameter d_1, defined in (14), change with the parameters μ, σ and r.

Proposition 8 *The parameter d_1 decreases with μ and σ and increases with r.*

Proof of Proposition 8 We look at d_1, defined in (14),

$$d_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \nabla,$$

with $\nabla = \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$. It is straightforward to conclude that d_1 increases with r. It is also very simple to take conclusions with respect to μ and σ, considering the derivatives, namely,

$$\frac{\partial d_1(\mu)}{\partial \mu} = -\frac{d_1}{\sigma^2\nabla} < 0 \quad \text{and} \quad \frac{\partial d_1(\sigma)}{\partial \sigma} = \frac{2}{\sigma^3\nabla}\left(\mu d_1 - r\right) < 0$$

because $r - \mu d_1 = \frac{\sigma^2}{2} d_1(d_1 - 1) > 0$. Thus, d_1 decreases with μ and σ.

■

6.2.1 Proof of Proposition 5

The proof for the benchmark case is trivial, as the equation that defines $x_B^\star$ is easy to analyze. Therefore we skip the proof. For the stochastic case, the calculations are less trivial and therefore we present them in here.

We have $x_S^\star = \frac{K_b}{K_1}$, where $K_b = \frac{d_1}{2(d_1-1)} \frac{(r-\mu)\pi_0}{r\beta K_0}$. As K_b does not depend on δ, λ and u, neither does $x_S^\star$. Also K_b does not depend on K_1, but $x_S^\star$ decreases with it. Obviously, K_b decreases with β and $x_S^\star$, as well. Taking into account the π_0 definition, we easily conclude that K_b increases with ξ_0 and decreases with α and K_0, and therefore so does $x_S^\star$. To study the behavior of $x_S^\star$ with σ, we need to explore $\frac{d_1(\sigma)}{d_1(\sigma)-1} = 1 + \frac{1}{d_1(\sigma)-1}$, as a function of σ. Given that d_1 decreases with σ, then $\frac{d_1(\sigma)}{d_1(\sigma)-1}$ increases with σ, as well as K_b and, consequently, $x_S^\star$. The challenging cases are μ and r.

For μ, we need to study $\varrho_1(\mu) = (r-\mu)\frac{d_1(\mu)}{d_1(\mu)-1}$. Note that $\varrho_1'(\mu) = -\frac{d_1(\mu)[d_1(\mu)-1]+(r-\mu)\frac{\partial d_1(\mu)}{\partial \mu}}{[d_1(\mu)-1]^2}$, which can be simplified as $\varrho_1'(\mu) = -\frac{d_1(\mu)}{\sigma^2 \nabla}\left[\sigma^2 \nabla \left(d_1(\mu)-1\right) - (r-\mu)\right]$. After some calculations we end up with $\varrho_1'(\mu) = -\frac{d_1(\mu)}{\nabla}\left[\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{r+\mu}{\sigma^2} - \left(\frac{1}{2} + \frac{\mu}{\sigma^2}\right)\nabla\right]$, to which we can apply the conjugate and after some comprehensive calculations, we get $\varrho_1'(\mu) = -\frac{d_1(\mu)\left(\frac{r-\mu}{\sigma^2}\right)^2}{\nabla\left[\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{r+\mu}{\sigma^2} + \left(\frac{1}{2}+\frac{\mu}{\sigma^2}\right)\nabla\right]} < 0$. Therefore, we conclude that K_b and $x_S^\star$ decrease with μ.

For r we need to study $\varrho_2(r) = \frac{r-\mu}{r}\frac{d_1(r)}{d_1(r)-1}$. Note that $\varrho_2'(r) = \frac{\mu d_1(r)(d_1(r)-1) - r(r-\mu)\frac{\partial d_1(r)}{\partial r}}{[r(d_1(r)-1)]^2}$, which is equivalent to $\varrho_2'(r) = \frac{2\mu\nabla\left[r-\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)-\mu\nabla\right]-r(r-\mu)}{\nabla[r\sigma(d_1(r)-1)]^2}$. Applying again the conjugate, we obtain $\varrho_2'(r) = -\frac{(r-\mu)^2}{\nabla[\sigma(d_1(r)-1)]^2\left[2\mu\nabla\left[r-\mu\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)\right]+\left[2\mu^2\left[\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]+r(r-\mu)\right]\right]} < 0$. Then, K_b and $x_S^\star$ also decrease with r.

■

6.2.2 Proof of Proposition 6

We want to investigate how the decision threshold, $\theta_S^\star$, defined in (19), evolves with the different parameters. The functions that define $\theta_S^\star$, presented in Proposition 2, are elementary functions of the parameters, and therefore their analysis is straightforward. For that reason, we omit these derivations.

■

6.2.3 Proof of Proposition 7

Contrary to the benchmark case, the stochastic case the proofs for the comparative statics of the investment threshold require many arguments, calculations, and we need to invoke an auxiliary lemma, that we provide (along with its proof) afterwards.

We start by recalling the expressions that define the investment thresholds curves for the stochastic case:

$$\begin{aligned} \upsilon_S^A(x) &= \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + 2\beta K_0 - \frac{a(r-\mu)x^{d_1-1}}{K_1} \\ \upsilon_S^R(x) &= \frac{\epsilon}{x} + \frac{\lambda u}{r} + \alpha K_1 + \frac{(r-\mu)\pi_0}{rK_1 x}. \end{aligned}$$

It is immediate to see that the derivatives of υ_S^A and υ_S^R in order to either δ, λ or u are equal and positive, and thus $\theta_S^\star$ increases with those parameters, and that the opposite holds for x.

Note that we can rewrite a as follows $a = \left[\frac{r(d_1-1)}{\pi_0}\right]^{d_1-1}\left[\frac{2\beta K_0 K_1}{(r-\mu)d_1}\right]^{d_1}$, showing that a decreases with ξ_0 and increases with β. Then, $\frac{\partial \upsilon_S^A(x,\xi_0)}{\partial \xi_0} = -\frac{(r-\mu)x^{d_1(\xi_0)-1}}{K_1}\frac{\partial a(\xi_0)}{\partial \xi_0} > 0$ and $\frac{\partial \upsilon_S^R(x,\xi_0)}{\partial \xi_0} = \frac{(r-\mu)K_0}{rK_1 x} > 0$, implying that $\theta_S^\star$ increases with ξ_0. Now we notice that υ_S^R does not depend on either σ or β, thus for replacement the threshold $\theta_S^\star$ is the same for any σ or β. The proofs for those two parameters are very similar.

Regarding σ, let us consider $\sigma_1 < \sigma_2$. First we want to discover the sign of $\upsilon_S^A(x,\sigma_2) - \upsilon_S^A(x,\sigma_1) = -\frac{r-\mu}{K_1}\left[a(\sigma_2)x^{d_1(\sigma_2)-1} - a(\sigma_1)x^{d_1(\sigma_1)-1}\right]$. We have that $\upsilon_S^A(x,\sigma_2) > \upsilon_S^A(x,\sigma_1) \Leftrightarrow x > \left[\frac{a(\sigma_2)}{a(\sigma_1)}\right]^{\frac{1}{d_1(\sigma_1)-d_1(\sigma_2)}}$, which means that the graphs of $\upsilon_S^A(x,\sigma_1)$ and $\upsilon_S^A(x,\sigma_2)$ intercept each other only once. We know that the graphs of $\upsilon_S^A(x,\sigma_1)$ and $\upsilon_S^A(x,\sigma_2)$ are tangent to the graph of $\upsilon_S^R(x)$, respectively, at $x_S^\star(\sigma_1)$ and $x_S^\star(\sigma_2)$, where we already proved that $x_S^\star(\sigma_1) < x_S^\star(\sigma_2)$. Given that $\upsilon_S^A(x_S^\star(\sigma_1),\sigma_1) = \upsilon_S^R(x_S^\star(\sigma_1)) > \upsilon_S^A(x_S^\star(\sigma_1),\sigma_2)$ and $\upsilon_S^A(x_S^\star(\sigma_2),\sigma_2) = \upsilon_S^R(x_S^\star(\sigma_2)) > \upsilon_S^A(x_S^\star(\sigma_2),\sigma_1)$, this implies that the graphs of $\upsilon_S^A(x,\sigma_1)$ and $\upsilon_S^A(x,\sigma_2)$ need to intercept each other between $x_S^\star(\sigma_1)$ and $x_S^\star(\sigma_2)$. Since they only intercept once, we conclude that $x_S^\star(\sigma_1) < \left[\frac{a(\sigma_2)}{a(\sigma_1)}\right]^{\frac{1}{d_1(\sigma_1)-d_1(\sigma_2)}} < x_S^\star(\sigma_2)$. We can conclude that $\theta_S^\star(x,\sigma_1) > \theta_S^\star(x,\sigma_2)$ for $0 < x < x_S^\star(\sigma_2)$ and $\theta_S^\star(x,\sigma_1) = \theta_S^\star(x,\sigma_2)$ for $x \geq x_S^\star(\sigma_2)$.

Concerning β, let us take $\beta_1 < \beta_2$, and study the sign of $\upsilon_S^A(x,\beta_2) - \upsilon_S^A(x,\beta_1) = 2(\beta_2-\beta_1)K_0 - \frac{(r-\mu)x^{d_1-1}}{K_1}[a(\beta_2) - a(\beta_1)]$. As in the previous case, the graphs of $\upsilon_S^A(x,\beta_1)$ and $\upsilon_S^A(x,\beta_2)$ only intercept each other once; also $\upsilon_S^A(x,\beta_2) > \upsilon_S^A(x,\beta_1) \Leftrightarrow x > \left[\frac{2(\beta_2-\beta_1)K_0K_1}{(r-\mu)[a(\beta_2)-a(\beta_1)]}\right]^{\frac{1}{d_1-1}}$[5]. Given that $\upsilon_S^A(x^\star(\beta_1),\beta_1) =$

[5] Given that a increases with β, we have $a(\beta_2) - a(\beta_1) > 0$.

$\upsilon_S^R\left(x_S^\star(\beta_1)\right) > \upsilon_S^A\left(x_S^\star(\beta_1),\beta_2\right)$ and $\upsilon_S^A\left(x_S^\star(\beta_2),\beta_2\right) = \upsilon_S^R\left(x_S^\star(\beta_2)\right) > \upsilon_S^A\left(x_S^\star(\beta_2),\beta_1\right)$, and recalling that $x_S^\star(\beta_2) < x_S^\star(\beta_1)$, then the graphs of $\upsilon_S^A\left(x,\beta_1\right)$ and $\upsilon_S^A\left(x,\beta_2\right)$ need to intercept each other between $x_S^\star(\beta_2)$ and $x_S^\star(\beta_1)$. Since they only intercept once, we conclude that $x_S^\star(\beta_2) < \left[\frac{2(\beta_2-\beta_1)K_0K_1}{(r-\mu)[a(\beta_2)-a(\beta_1)]}\right]^{\frac{1}{d_1-1}} < x_S^\star(\beta_1)$. We can conclude that $\theta_S^\star(x,\beta_1) < \theta_S^\star(x,\beta_2)$ for $0 < x < x_S^\star(\beta_1)$ and $\theta_S^\star(x,\beta_1) = \theta_S^\star(x,\beta_2)$ for $x \geq x_S^\star(\beta_1)$.

The study of μ has some more complex details. The replace case is straightforward, as $\frac{\partial \upsilon_S^R(x,\mu)}{\partial \mu} = -\frac{1}{x}\left[\delta + \frac{\pi_0(\mu)}{rK_1}\right] < 0$. Let us consider $\upsilon_S^A(x,\mu_2) - \upsilon_S^A(x,\mu_1) = -\frac{1}{x}\nu(x)$, where

$$\nu(x) = \delta\left(\mu_2-\mu_1\right) + \frac{1}{K_1}\left[a(\mu_2)(r-\mu_2)x^{d_1(\mu_2)} - a(\mu_1)(r-\mu_1)x^{d_1(\mu_1)}\right],$$

for $\mu_2 > \mu_1$. Given that $d_1(\mu_1) > d_1(\mu_2)$, we have $\nu(0) = \delta\left(\mu_2-\mu_1\right) > 0$ and $\lim_{x\to+\infty}\nu(x) = -\infty$. Considering the derivative

$$\nu'(x) = \frac{1}{K_1}\left[a(\mu_2)(r-\mu_2)d_1(\mu_2)x^{d_1(\mu_2)-1} - a(\mu_1)(r-\mu_1)d_1(\mu_1)x^{d_1(\mu_1)-1}\right],$$

we see that its sign is changing only once; indeed $\nu'(x) > 0 \Leftrightarrow x < \left[\frac{a(\mu_2)(r-\mu_2)d_1(\mu_2)}{a(\mu_1)(r-\mu_1)d_1(\mu_1)}\right]^{\frac{1}{d_1(\mu_1)-d_1(\mu_2)}}$. Then ν has only one zero, changing from positive to negative. Moreover, given that υ_S^R decreases on μ, we have $\upsilon_S^A(x_S^\star(\mu_2),\mu_1) > \upsilon_S^R(x_S^\star(\mu_2),\mu_1) > \upsilon^R(x_S^\star(\mu_2),\mu_2) = \upsilon_S^A(x_S^\star(\mu_2),\mu_2)$, which means that $\nu\left(x_S^\star(\mu_2)\right) > 0$. Thus, for $0 < x \leq x_S^\star(\mu_2)$, we have $\nu(x) > 0$, i.e. $\upsilon_S^A(x,\mu_2) < \upsilon_S^A(x,\mu_1)$. It remains to see that for $x_S^\star(\mu_2) < x \leq x_S^\star(\mu_1)$, $\upsilon_S^R(x,\mu_2) < \upsilon_S^R(x,\mu_1) \leq \upsilon_S^A(x,\mu_1)$. With this we conclude that $\theta_S^\star$ decreases with μ.

The behaviour of $\theta_S^\star$ with respect to K_0, K_1, α and r depends intrinsically on the way the functions υ_S^A and υ_S^R behave when one changes values of the parameters. As some of the arguments are similar for all these parameters, we propose to use y to denote one of the parameters under study.

In order to clarify the proof, we present a sketch of the basic idea. We start by recalling that $x_S^\star$ decreases with K_0, K_1, α and r, i.e., for $y_1 < y_2$, we have $x_S^\star(y_1) > x_S^\star(y_2)$. In all cases, the following functions have to be compared

$$\Phi_{AA}(x;y_1,y_2) = \upsilon_S^A(x,y_2) - \upsilon_S^A(x,y_1) \tag{24}$$

$$\Phi_{RR}(x;y_1,y_2) = \upsilon_S^R(x,y_2) - \upsilon_S^R(x,y_1) \tag{25}$$

$$\Phi_{RA}(x;y_1,y_2) = \upsilon_S^R(x,y_2) - \upsilon_S^A(x,y_1) \tag{26}$$

In Lemma 1 we derive the following expressions

$$\Phi_{AA}(x;y_1,y_2)=\begin{cases}\varsigma_1-\varsigma_2x^{d_1-1} & \text{if the parameter is } \alpha, K_0 \text{ or } K_1\\ \frac{1}{x}\left[\varsigma_4-\varsigma_5x^{d_1(y_2)}+\varsigma_6x^{d_1(y_1)}\right]-\varsigma_3 & \text{if the parameter is } r\end{cases} \tag{27}$$

where $\varsigma_i>0$ for $i=1,2,3,4,5,6$. Furthermore,

$$\Phi_{RR}(x;y_1,y_2)=\varsigma_1-\frac{\varsigma_2}{x} \tag{28}$$

where, for $i=1,2$, $\varsigma_i>0$ for K_1 and α, and $\varsigma_i<0$ for r. For K_0, $\varsigma_1=0$ and $\varsigma_2>0\Leftrightarrow y_1+y_2<\frac{\xi_0}{\alpha}$. Finally,

$$\Phi_{RA}(x;y_1,y_2)=\frac{\varsigma_1}{x}\left[\varsigma_2x^{d_1}+\varsigma_3\right]-\varsigma_4 \tag{29}$$

where $\varsigma_i>0$ for $i=1,2,3$ and $\varsigma_4\in\mathbb{R}$.

Next we study the sign of these functions. When the parameter in study is α, K_0 or K_1, then $\Phi_{AA}(x;y_1,y_2)>0\Leftrightarrow x<\left(\frac{\varsigma_1}{\varsigma_2}\right)^{\frac{1}{d_1-1}}=i_1$. For r, as $\lim_{x\to0^+}\Phi_{AA}(x;y_1,y_2)=+\infty$ and $\lim_{x\to+\infty}\Phi_{AA}(x;y_1,y_2)=-\infty$, it follows that Φ_{AA} has at least one zero. So, here we just consider the case where we have exactly one zero, hereby denoted by i_1. Then $\Phi_{AA}(x;y_1,y_2)>0\Leftrightarrow x<i_1$.

For Φ_{RR} the following holds:

- for K_0, $\Phi_{RR}(x;y_1,y_2)>0\Leftrightarrow y_1+y_2<\frac{\xi_0}{\alpha}$;
- for K_1 and α, $\Phi_{RR}(x;y_1,y_2)>0\Leftrightarrow x>\frac{\xi_2}{\xi_1}=i_2$;
- for r, $\Phi_{RR}(x;y_1,y_2)>0\Leftrightarrow x<\frac{\xi_2}{\xi_1}=i_2$.

Regarding Φ_{RA}, given that $d_1>1$, then it follows that $\lim_{x\to0^+}\Phi_{RA}(x;y_1,y_2)=+\infty$ and $\lim_{x\to+\infty}\Phi_{RA}(x;y_1,y_2)=+\infty$. We have $\Phi'_{RA}(x;y_1,y_2)=\frac{\varsigma_1}{x^2}\left[\varsigma_2(d_1-1)x^{d_1}-\varsigma_3\right]$, which has only one zero when $x=\left[\frac{\varsigma_3}{\varsigma_2(d_1-1)}\right]^{\frac{1}{d_1}}$. Thus Φ_{RA} is a convex function, that either it is always non negative or there are two points, j_1 and j_2, such that $\Phi_{RA}(x;y_1,y_2)<0\Leftrightarrow x\in(j_1,j_2)$. This fact only depends on the sign of $m=\Phi_{RA}\left(\left[\frac{\varsigma_3}{\varsigma_2(d_1-1)}\right]^{\frac{1}{d_1}};y_1,y_2\right)=d_1\varsigma_1\varsigma_2^{\frac{1}{d_1}}\left[\frac{\varsigma_3}{d_1-1}\right]^{\frac{d_1-1}{d_1}}-\varsigma_4$. Summing up, if $m\geq0$ then $\Phi_{RA}(x;y_1,y_2)\geq0$ for all $x>0$; if $m<0$ then $\Phi_{RA}(x;y_1,y_2)<0\Leftrightarrow x\in(j_1,j_2)$.

In view of the behaviour of these functions, for α and K_1 we conclude that: if $m\geq0$ then $\theta^\star_S(x,y_1)\leq\theta^\star_S(x,y_2)$; if $m<0$ then $\theta^\star_S(x,y_1)\leq\theta^\star_S(x,y_2)$ when $0<x\leq\varpi_1$ or $x\geq\varpi_2$ and $\theta^\star_S(x,y_1)>\theta^\star_S(x,y_2)$ when $\varpi_1<x<\varpi_2$, where

- if $i_1 \leq x_S^\star(y_2)$ then $\varpi_1 = i_1$, otherwise $\varpi_1 = j_1$;

- if $i_2 \geq x_S^\star(y_1)$ then $\varpi_2 = i_2$, otherwise $\varpi_2 = j_2$.

For K_0, if $y_1 + y_2 \leq \frac{\xi_0}{\alpha}$ then $\theta_S^\star(x, y_1) \leq \theta_S^\star(x, y_2)$; otherwise $\theta_S^\star(x, y_1) \leq \theta_S^\star(x, y_2)$ when $0 < x \leq \varpi$ and $\theta_S^\star(x, y_1) > \theta_S^\star(x, y_2)$ when $x > \varpi$, where $\varpi = i_1$ if $i_1 \leq x_S^\star(y_2)$ and $\varpi = j_1$ if $i_1 > x_S^\star(y_2)$.

Finally, with respect to r, we have the following behaviours: $\theta_S^\star(x, y_1) \leq \theta_S^\star(x, y_2)$ when $0 < x \leq \varpi$ and $\theta_S^\star(x, y_1) > \theta_S^\star(x, y_2)$ when $x > \varpi$, where

- $\varpi = i_1$ if $m > 0$ and $j_1 < x_S^\star(y_2) < x_S^\star(y_1) < j_2$;

- $\varpi = j_1$ if $m > 0$ and $x_S^\star(y_2) < x_S^\star(y_1) < j_2$;

- $\varpi = i_2$ otherwise [6].

■

Lemma 1 *For the parameters K_0, K_1, α and r, the functions defined in* (24), (25) *and* (26) *are given by* (27), (28) *and* (29), *respectively.*

Proof of Lemma 1

- For parameter K_0, let us consider two possible values, $K_{01} < K_{02}$.

  i) $\Phi_{AA}(x; K_{01}, K_{02}) = 2\beta\left(K_{02} - K_{01}\right) - \frac{(r-\mu)[a(K_{02}) - a(K_{01})]}{K_1} x^{d_1 - 1}$.

     Note that $a\left(K_{02}\right) - a\left(K_{01}\right) > 0 \Leftrightarrow K_{01}\left[\xi_0 - \alpha K_{01}\right]^{1-d_1} < K_{02}\left[\xi_0 - \alpha K_{02}\right]^{1-d_1}$. Considering the function $\phi(k) = k\left(\xi_0 - \alpha k\right)^{1-d_1}$, with $k < \frac{\xi_0}{\alpha}$, we have $\phi'(k) = \frac{(\xi_0 - \alpha k) + \alpha k (d_1 - 1)}{(\xi_0 - \alpha k)^{d_1}} > 0$, meaning that ϕ is an increasing function. Thus, it holds $a\left(K_{02}\right) - a\left(K_{01}\right) > 0$.

  ii) $\Phi_{RR}(x; K_{01}, K_{02}) = \frac{(r-\mu)[\pi_0(K_{02}) - \pi_0(K_{01})]}{r K_1 x}$.

     Note that $\pi_0\left(K_{02}\right) - \pi_0\left(K_{01}\right) > 0 \Leftrightarrow K_{01} + K_{02} < \frac{\xi_0}{\alpha}$.

  iii) $\Phi_{RA}(x; K_{01}, K_{02}) = \frac{(r-\mu)}{r K_1 x}\left[r a\left(K_{01}\right) x^{d_1} + \pi_0\left(K_{02}\right)\right] - 2\beta K_{01}$.

- For parameter K_1, let us consider two possible values, $K_{11} < K_{12}$.

[6] Note that we always have $x_S^\star(y_1) < j_2$

i) $\Phi_{AA}(x; K_{11}, K_{12}) = \alpha\left(K_{12} - K_{11}\right) - \frac{(r-\mu)[K_{11}a(K_{12}) - K_{12}a(K_{11})]}{K_{11}K_{12}} x^{d_1 - 1}$.

Note that $K_{11}a(K_{12}) - K_{12}a(K_{11}) > 0 \Leftrightarrow \left(\frac{K_{11}}{K_{12}}\right)^{d_1-1} < 1$, which is true because $d_1 > 1$ and $\frac{K_{11}}{K_{12}} < 1$.

ii) $\Phi_{RR}(x; K_{11}, K_{12}) = (K_{12} - K_{11})\left[\alpha - \frac{(r-\mu)\pi_0}{rK_{11}K_{12}x}\right]$.

iii) $\Phi_{RA}(x; K_{11}, K_{12}) = \frac{(r-\mu)}{x}\left[\frac{a(K_{11})}{K_{11}} x^{d_1} + \frac{\pi_0}{rK_{12}}\right] - [2\beta K_0 - \alpha\left(K_{12} - K_{11}\right)]$.

- For parameter α, let us consider two possible values, $\alpha_1 < \alpha_2$.

i) $\Phi_{AA}(x; \alpha_1, \alpha_2) = K_1\left(\alpha_2 - \alpha_1\right) - \frac{(r-\mu)[a(\alpha_2) - a(\alpha_1)]}{K_1} x^{d_1-1}$.

Note that $a(\alpha_2) - a(\alpha_1) = \left[\frac{2\beta K_0 K_1}{(r-\mu)d_1}\right]^{d_1} \left[r(d_1-1)\right]^{d_1-1} \left[\pi_0(\alpha_2)^{1-d_1} - \pi_0(\alpha_1)^{1-d_1}\right]$ and $\pi_0(\alpha_2)^{1-d_1} - \pi_0(\alpha_1)^{1-d_1} > 0 \Leftrightarrow \left[\frac{\xi_0 - \alpha_1 K_0}{\xi_0 - \alpha_2 K_0}\right]^{d_1-1} > 1$, which is always true because $\frac{\xi_0 - \alpha_1 K_0}{\xi_0 - \alpha_2 K_0} > 1$ and $d_1 > 1$.

ii) $\Phi_{RR}(x; \alpha_1, \alpha_2) = (\alpha_2 - \alpha_1)\left[K_1 - \frac{(r-\mu)K_0{}^2}{rK_1 x}\right]$.

iii) $\Phi_{RA}(x; \alpha_1, \alpha_2) = \frac{(r-\mu)}{rK_1 x}\left[ra(\alpha_1)x^{d_1} + \pi_0(\alpha_2)\right] - [2\beta K_0 - K_1(\alpha_2 - \alpha_1)]$.

- For parameter r, let us consider two possible values, $r_1 < r_2$.

i) $\Phi_{AA}(x; r_1, r_2) = \frac{1}{x}\left[\delta(r_2 - r_1) - \frac{1}{K_1}\left[(r_2-\mu)a\left(r_2\right)x^{d_1(r_2)-1} - (r_1-\mu)a\left(r_1\right)x^{d_1(r_1)-1}\right]\right] + \frac{\lambda u(r_2 - r_1)}{r_1 r_2}$.

ii) $\Phi_{RR}(x; r_1, r_2) = \frac{(r_2 - r_1)}{r_1 r_2}\left[-\lambda u + \frac{\delta K_1 r_1 r_2 + \mu\pi_0}{K_1 x}\right]$.

iii) $\Phi_{RA}(x; r_1, r_2) = \frac{1}{r_2 K_1 x}\left[r_2(r_1-\mu)a(r_1)x^{d_1(r_1)} + r_2\left(r_2 - r_1\right)\delta K_1 + (r_2-\mu)\pi_0\right] - \left[2\beta K_0 + \frac{\lambda u(r_2 - r_1)}{r_1 r_2}\right]$.

■

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